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10 February 1993

Dr. Peter Wakker
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Dear Peter Wakker:

The 1941 text in the Harvard records is exactly the same as the 1947 text. To illustrate the point I have photocopied representative pages: you will notice that Equations (1)-(33) of *Foundations*, pp. 174-181, are precisely (1)-(33) on pages 151-157 of the November, 1940 Harvard Ph.D. submission. The footnotes and text wordings also agree. [At some later chapters the two texts do differ more.]

While working during World War II at the Radiation Laboratory (as a mathematician designing anti-aircraft servo systems) I must have added the useful last paragraph of page 179 with its insightful equation (34)--which I believe I arrived at c.1937. From then on the texts pretty much agree. (A Harvard professor of famous stupidity held up publication for some years through sheer inadvertence or Freudian resistance.)

Sincerely,

Paul A Samuelson

Paul A. Samuelson

PAS/jmm

P.S. The reference to Milton Friedman's maiden paper aroused memories. Keynes refused it because Pigou correctly asserted that M.F. wrongly criticized him. For years I thought it ought to have been published if more guardedly written; but I had quite forgotten my validation of Pigou's defense!

With the Compliments

of

PAUL A. SAMUELSON

Enclosed with Samuelson's letter of February 10, 1953.

FOUNDATIONS OF ANALYTICAL ECONOMICS

The Observational Significance of Economic Theory

by

Paul A. Samuelson

"Mathematics is a Language"

J. Willard Gibbs

Nov. '40

PREFACE

The bulk of the present manuscript was conceived and written in 1937, but the pressure of other interests delayed my bringing it into final form. It consists entirely of hitherto unpublished material, although in one of the chapters I lean heavily upon some of my publications in various academic journals. Fortunately, the passage of time has dealt kindly with the analysis contained here, and where it abuts upon the topics treated in Professor Hicks' long awaited treatise, the similarity in point of view has been reassuring.

My greatest debt is to Marion Crawford Samuelson whose suggestions and corrections have been all too many. The result has been a vast mathematical, economic, and stylistic improvement, and no perfunctory acknowledgment can bear adequate witness to her contribution. My thanks for prolonged stimulation over many years must go out to Professors Schumpeter, Leontief, and E. B. Wilson, while each of a legion of Harvard graduate students has left his mark upon what follows. Finally, grateful acknowledgment is made to the Social Science Research Council and to the Society of Fellows of Harvard University for the opportunities they provided for pursuit of independent research.

Cambridge, Massachusetts
November, 1940

CHAPTER VI

SPECIAL ASPECTS OF THE THEORY OF CONSUMER'S BEHAVIOR

The last chapter exhausts the content of the utility analysis in its most general form, involving only an ordinal preference field. There remains in the literature a great number of discussions of particular problems which involve special and extra assumptions. In order to present a fairly complete account of the present status of the theory I propose to examine some of these carefully to show their empirical meaning. This involves a break in the unity of exposition since each special assumption has often been made independently of all others. There is no choice but to go through the list with no regard for continuity. Among the topics to be discussed will be the cardinal measure of utility, independence of utilities and measures of complementarity, and constancy of the marginal utility of money. Excluded because of lack of space are the questions of "integrability" and the economic theory of index numbers.

It is clear that every assumption either places restrictions upon our empirical data or is meaningless. A price must be paid for any simplifications introduced into our basic hypotheses. This price is the limiting of the field of applicability and relevance of the theory because of the extra empirical restrictions to be imposed on the data. Many writers do not appear to be only two consumption goods are involved.

Given a great number of observations on prices, quantities, and total income, one could in the limit more or less trace out the whole indifference map. We should still, however, have said nothing about the numbering of the one parameter family of indifference loci so traced out. Both Professors Frisch and Fisher employ the following definition for selecting out a particular utility index to be designated the "true" measure of utility, subject to origin and scale constants. That utility index, if it exists, is to be selected which can be written in the form

$$\varphi = f(x) + g(y) \quad ; \tag{1}$$

i.e., for which

$$\frac{\partial^2 \varphi}{\partial x \, \partial y} = 0 \quad . \tag{2}$$

If there exists one index of utility which can be written in the form of equation (1), then any other index which obeys the same law must differ from it only by a linear transformation. For, consider another index

$$F = F(\varphi) \quad , \tag{3}$$

for which

$$\frac{\partial^2 F}{\partial x \, \partial y} = 0 \quad . \tag{4}$$

Successively differentiating (3) partially with respect to $\underline{x}$ and $\underline{y}$, we get

$$\frac{\partial^2 F}{\partial x \, \partial y} = F' \varphi_{xy} + F'' \varphi_x \varphi_y \quad . \tag{5}$$

This, together with (2) requires that

$$F''\varphi_x\varphi_y \equiv 0 \quad , \tag{6}$$

or

$$F''(\varphi) \equiv 0 \quad . \tag{7}$$

Therefore,

$$F = a + b\varphi \quad , \tag{8}$$

where a and b are origin and scale constants respectively.

It is clear, therefore, that the assumption that utilities shall be "independent" will help to select one utility index as the cardinal measure of utility. Nevertheless, even this convention is not in general applicable. It will guarantee us that we do not have two different utility scales, as has been shown in the above proof; it will not, in general, provide us with even one scale.

If we assume an indifference field obeying the ordinary concavity restrictions and nothing more, then there will not, in general, be even one utility index which can be written in the form

$$\varphi = f(x) + g(y) \quad , \tag{9}$$

$$\varphi_{xy} \equiv 0 \quad .$$

Let us write out one legitimate utility index.

$$H = H(x,y) \quad . \tag{10}$$

Does there exist a transformation F such that

$$\varphi = F\left[H(x,y)\right] = f(x) + g(y) \quad ? \tag{11}$$

The answer in general must be in the negative. Further arbitrary restrictions must first be placed upon the indifference field.

Let the indifference field be defined in the following form, independent of any utility concept:

$$-\left(\frac{dy}{dx}\right)_{H\,=\,\text{constant}} = R(x,y) \quad , \tag{12}$$

where R is a function of $\underline{x}$ and $\underline{y}$ obeying the following curvature requirement:

$$R_x - R_y R < 0 \quad . \tag{13}$$

The necessary and sufficient condition that there exist a utility index which can be written in the form

$$\varphi = f(x) + g(y) \quad ,$$

$$\varphi_{xy} \equiv 0$$

is as follows:

$$R_{xy} - R_x R_y \equiv 0 \quad , \tag{14}$$

or

$$\frac{\partial^2 \log R}{\partial x\, \partial y} \equiv 0 \quad . \tag{15}$$

The necessity is verified by the differentiation of

$$R(x,y) = \frac{f(x)}{g(y)} \quad . \tag{16}$$

The sufficiency is also easily indicated.

If

$$\frac{\partial^2 \log R}{\partial x\, \partial y} \equiv 0 \quad ,$$

$$\log R = \log h(x) - \log k(y) = \log \frac{h(x)}{k(y)} \quad . \tag{17}$$

where h and k are arbitrary functions. Form the expression

$$R\,dx + dy = \frac{h(x)}{k(y)}\,dx + dy \quad . \tag{18}$$

This can easily be transformed into the exact differential

$$dQ = h(x)\,dx + k(y)\,dy \quad , \tag{19}$$

or

$$\begin{aligned} Q &= \int_a^x h(x)\,dx + \int_b^y k(y)\,dy + \text{constant} \\ &= f(x) + g(y) \quad . \end{aligned} \tag{20}$$

We must now investigate the meaning of the restriction in (14). The assumption of independence of utilities in order to define a cardinal measure of utility is seen to involve (1) a convention by means of which one out of an infinity of possible utility scales is designated as the true cardinal measure of utility; (2) an arbitrary *a priori* restriction upon the preference field, and hence upon empirical price-quantity behavior. The meaning of this restriction we must now investigate.

The functional restriction (14) is a partial differential equation of the second order of the general form

$$M(R, R_x, R_y, R_{xx}, R_{xy}, R_{yy}, x, y) = 0 \quad . \tag{21}$$

Subject to boundary conditions involving two arbitrary functions, it will serve to define a unique solution function

$$R = \overline{R}(x, y) \quad . \tag{22}$$

More specifically, if we are given as empirical observational data the two expenditure paths corresponding to the changes in quantities with income in each of two respective price situations, then from these observations, and these alone, the whole field of indifference curves can be determined by suitable extrapolation.

It is not easy to visualize intuitively why this should be so; indeed, few economists would be so bold as to claim that the behavior of an individual in all conceivable circumstances should be derivable from so few observations. And yet this is the conclusion to which we are forced by the apparently innocuous assumption of independence of utilities.

Moreover, (14) places definite restrictions on our demand functions, the validity of which are equally dubious and equally impossible to comprehend intuitively. For the simple two commodity case our conditions of demand equilibrium can be written

$$\frac{p_x}{p_y} = R(x,y) \quad ,$$
$$I = p_x x + p_y y \quad . \tag{23}$$

These may be transformed into

$$x = m\left(\frac{p_x}{p_y}, \frac{I}{p_y}\right) \quad ,$$
$$y = n\left(\frac{p_x}{p_y}, \frac{I}{p_y}\right) \tag{24}$$

These demand equations must be subject to the restriction

$$\frac{\partial^2 \log \frac{p_x}{p_y}}{\partial x \, \partial y} = 0 \quad . \tag{25}$$

When there are more than two goods, the restrictions implied by the very possibility of an independent index of utility take on a different and more complicated form. If there exists an index F for which

$$F_{ij} \equiv 0 \quad , \qquad i \neq j \tag{26}$$

then

$$F_{ij} = F'(\varphi)\varphi_{ij} + F''(\varphi)\varphi_i\varphi_j \equiv 0 \quad , \tag{27}$$

where φ is any other index. Thus,

$$\frac{\varphi_{ij}}{\varphi_i\varphi_j} \equiv T(\varphi) \quad , \qquad i \neq j \tag{28}$$

where T is an arbitrary function and φ is any index of utility. Taking into account the $\frac{(n-1)(n-2)}{2}$ conditions of integrability, this implies an additional $\frac{n(n-1)}{2}$ conditions. It is to be noted that these are identities, holding everywhere. Not only are they necessary, but the transformation

$$F = \int_a^{\varphi} e^{\int_c^v T(\theta)d\theta} dv \tag{29}$$

shows them to be sufficient as well.

In terms of the indifference varieties these take still another form. Let

$$-\frac{p_j}{p_i} = \frac{dx_i}{dx_j} = {}^iR^j(x_1,\ldots,x_n) \quad . \tag{30}$$

These, of course, satisfy the identities

$$\frac{{}^iR^j}{{}^iR^k} \equiv {}^kR^j \quad . \qquad (i,j,k = 1,\ldots,n) \tag{31}$$

If an independent index of utility is possible, then we must have

$$\frac{\partial\, {}^iR^j}{\partial x_k} \equiv 0 \quad . \qquad j \neq k \neq i \tag{32}$$

In view of the n^3 relations of (31), the above $n(n-1)^2$ relations are not all independent. At most $n(n-2)$ are independent, and these may be written in the form

$$\frac{\partial\ {}^1R^i}{\partial x_j} = 0 \quad , \quad i \neq j \quad ; \quad (i,j = 2,\ldots,n)$$

$$\frac{\partial\left(\frac{{}^1R^i}{{}^1R^2}\right)}{\partial x_1} = 0 \quad . \quad (i = 3,\ldots,n) \qquad (33)$$

These conditions are both necessary and sufficient. They imply among other things the $\frac{(n-1)(n-2)}{2}$ integrability conditions of equation (14) of the previous chapter. On the other hand, if the latter are postulated at the beginning, then equations (33) cease to be all independent and can be reduced in number.

We have then $n(n-2)$ partial differential equations of the first order. Subject to an equal number of arbitrary functions, the <u>general</u> solution is uniquely determined. But empirically an observation of an expenditure path involves $(n-1)$ functions. Hence, observation of more than $\frac{n(n-2)}{(n-1)}$, or more than n expenditure paths, could be used to disprove the possibility of independence.

Except for R. G. D. Allen's derivation of equation (15) for the two good case, I am not aware that these full price-quantity implications of independence have previously been derived.[7] However, fragmentary sets of necessary but by no means sufficient conditions for independent goods have been derived by Slutsky and

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[7] R. G. D. Allen, "A Comparison between Different Definitions of Complementary and Competitive Goods," <u>Econometrica</u>, Vol. II (1934), pp. 168-175.